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Multiplicities of monomial ideals

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Introduction

Let $S = K[x_1 \cdots x_n]$ be a polynomial ring over a field K with standard grading, $I \subset S$ a graded ideal. The multiplicity of S/I can be computed from a graded S-free resolution of S/I.

Let $h =$ height of I and let

$$\begin{aligned}\mathbf{F}: 0 \to \sum_j S^{\beta_{mj}}(-j) \to \cdots \to \sum_j S^{\beta_{ij}}(-j) \\ \to \sum_j S^{\beta_{(i-1)j}}(-j) \to \cdots \to \sum_j S^{\beta_{1j}}(-j) \to S \to S/I\end{aligned}$$

be a graded resolution of S/I. Then the multiplicity $e(S/I)$ is given by

$$\sum_j \sum_i (-1)^i \beta_{ij} j^k = \begin{cases} 0, & \text{for } 1 \leqslant k < h, \\ (-1)^h h! e(S/I), & \text{for } k = h. \end{cases}$$

This formula does not require the resolution to be minimal. Suppose, however, that $\mathbf{F}$ is the minimal resolution of S/I and let $m_i = \min\{j\colon \beta_{ij} \neq 0\}$ and $M_i = \max\{j\colon \beta_{ij} \neq 0\}$ be the minimal and maximal shifts in the ith term of the resolution.

A conjecture of Herzog, Huneke, and Srinivasan [2] states that the multiplicity of S/I is bounded above by $(1/h!) \prod_{i=1}^h M_i$. Moreover, when S/I is Cohen–Macaulay, or when

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$m = h$, $e(S/I)$ is conjectured to be bounded below by the products of the minimal shifts divided by $h!$. This conjecture has been proved for the stable monomial ideals and the squarefree strongly stable monomial ideals [2]. Even though it has been known for several other special cases [5], it is still open in general.

There is another natural, usually non-minimal, resolution for monomial ideals defined by D. Taylor, called Taylor resolution [3]. Suppose $\mathbf{F}$ is the Taylor resolution and let $L_i = \max\{j\colon \beta_{ij} \neq 0\}$ be the maximal shifts in the Taylor resolution of S/I. Then it is clear that $L_i \geqslant M_i$ for all i. For monomial ideals, we introduce a new bound for the multiplicity of S/I which we call the Taylor bound. It is $(1/h!)\prod_{i=1}^{h} L_i$, in analogy to the conjectured bound of Herzog, Huneke, and Srinivasan.

We prove (Corollary 4.3) this bound holds when S/I has codimension 2. Our proof consists of two steps. We first reduce the problem to the one on squarefree monomial ideals. Then we rephrase the problem in the language of antichains and then do an extensive counting and estimation to arrive at the desired bound. We also establish the Taylor bound for monomial ideals defined by almost complete intersection, see Theorem 5.2.

In Section 3.2 we conjecture that for a monomial ideal I the multiplicity of S/I is always bounded above by the Taylor bound. The first general case where it is yet to be proved is in codimension three. We note that it is true for codimension three Gorenstein algebras because we have the stronger upper bound in for these algebras. This is proved in [2] for general graded ideals.

Notice that our conjecture on the Taylor bound is equivalent to the following conjecture which is phrased in purely combinatorial terms. Stated in the language of antichains, our conjecture will read as follows: we denote the cardinality of a set A by $|A|$. Let $[n]$ denote the set of first n positive integers. Suppose $\mathcal{A}$ is an antichain on n vertices, that is, a collection of subsets of $[n]$ such that for any two elements $A, A' \in \mathcal{A}$ with $A \neq A'$ one has $A \not\subset A'$. We denote by $G(\mathcal{A})$ the set of generators of $\mathcal{A}$, that is, the set of minimal vertex covers of $\mathcal{A}$. Recall that a subset $B \subset [n]$ is a minimal vertex cover of $\mathcal{A}$, if $B \cap A \neq \emptyset$ for all $A \in \mathcal{A}$, and for any proper subset $C \subset B$ there exists $A' \in \mathcal{A}$ such that $C \cap A' = \emptyset$.

Our terminology is motivated by the fact that if $\mathcal{A}$ is an antichain, $P_A = \sum_{i\in A} x_i S$ for $A \in \mathcal{A}$, and $J = \bigcap_{A\in\mathcal{A}} P_A$, then J is a squarefree monomial ideal which is minimally generated by the monomials $x_B = \prod_{j\in B} x_j$ with $B \in G(\mathcal{A})$.

Denote by $h = h(\mathcal{A})$ the least cardinality of an element of $\mathcal{A}$ and by $e(\mathcal{A})$ the number of subsets in $\mathcal{A}$ of cardinality h. Let $G(\mathcal{A}) = \{G_1, \dots, G_m\}$. For a subset I of $[m]$, define $G_I = \bigcup_{i\in I} G_i$ and let $L_i = \max\{|G_I|\colon |I| = i\}$. Then we conjecture that $e(\mathcal{A}) \leqslant (1/h!)\prod_{i=1}^{h} L_i$.

Thus, we have proved this when $h(\mathcal{A}) = 2$, or when the cardinality of $G(\mathcal{A})$ is h or $h+1$. The first open case is when each element of $\mathcal{A}$ has cardinality at least 3.

In the first section of the paper, we prove a general bound for the multiplicity of monomial ideals. This has the advantage of being true for all monomial ideals but is a rather weak bound in general. Then we describe our terminology of antichains and the ideas that we need later in the paper. The third section sets up the Taylor bound and the reduction to squarefree monomial ideals. We also prove the Taylor bound for a class of generic monomial ideals in this section. Section 4 is completely devoted to proving the Taylor bound in codimension two and the last section is the proof of the bound for almost complete intersections.

1. A general bound for the multiplicity

Let K be a field, $S = K[x_1, \ldots, x_n]$ the polynomial ring in n variables over K. For a monomial $f \in S$, $f = x_1^{a_1} \cdots x_n^{a_n}$, we set $\deg_{x_i}(f) = a_i$ and if I is monomial ideal we denote by $G(I)$ the unique minimal set of monomial generators of I. Recall that there is a unique irredundant intersection $I = \bigcap_i I_i$, where each I_i is an ideal generated by powers of variables. The ideals I_i are called the *irreducible components of* I_i. Throughout this paper, $[n]$ will denote the set $\{1, 2, \ldots, n\}$ containing n elements.

The purpose of this section is to prove the following general bound for the multiplicity $e(S/I)$ of S/I.

Theorem 1.1. *Let I be a monomial ideal of codimension h with $G(I) = \{f_1, \ldots, f_m\}$. For $i = 1, \ldots, n$ we set $a_i = \max\{\deg_{x_i}(f_j)\colon\ j = 1, \ldots, m\}$. Assume that $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$. Then*

$$e(S/I) \leqslant \frac{1}{h!} \prod_{i=1}^{h} (a_i + a_{i+1} + \cdots + a_n).$$

Proof. Let $P_1, \ldots, P_r$ be the associated prime ideals of I of codimension h. Then P_i is of the form $(x_{i_1}, \ldots, x_{i_h})$ and IS_{P_i} is monomial ideal which is primary to $P_i S_{P_i}$. Therefore, IS_{P_i} contains pure powers of the x_{i_j}, say $IS_{P_i} = (x_{i_1}^{b_1}, \ldots, x_{i_h}^{b_h}, \ldots)S_{P_i}$. Since IS_{P_i} is a localization of I it is clear that $b_j \leqslant a_{i_j}$ for $j = 1, \ldots, h$. Thus, it follows that $\ell(S_{P_i}/IS_{P_i}) \leqslant a_{i_1} a_{i_2} \cdots a_{i_h}$.

Hence, the associativity formula for multiplicities [1] yields

$$e(S/I) = \sum_{i=1}^{r} \ell\big(S_{P_i} \big/ I_i S_{P_i}\big) \leqslant \sum_{1 \leqslant i_1 < i_2 < \cdots < i_h \leqslant n} a_{i_1} a_{i_2} \cdots a_{i_h}. \tag{1}$$

On the other hand, set $a_\sigma = \prod_{i \in \sigma} a_i$ for $\sigma \subset [n]$. We will show by induction on h that

$$\frac{1}{h!} \prod_{i=1}^{h} (a_i + a_{i+1} + \cdots + a_n) \geqslant \sum_{\sigma,\, |\sigma| = h} a_\sigma.$$

For $h = 1$, the assertion is trivial. So let $h > 1$. Then our induction hypothesis implies

$$\begin{aligned}
\frac{1}{(h-1)!} \prod_{i=1}^{h} (a_i + a_{i+1} + \cdots + a_n) &\geqslant (a_1 + \cdots + a_n) \Bigg(\sum_{\substack{\sigma \subset [2,n] \\ |\sigma| = h-1}} a_\sigma \Bigg) \\
&= a_1 \Bigg(\sum_{\substack{\sigma \subset [2,n] \\ |\sigma| = h-1}} a_\sigma \Bigg) + \sum_{i=2}^{n} a_i \Bigg(\sum_{\substack{\sigma \subset [2,n] \\ |\sigma| = h-1}} a_\sigma \Bigg),
\end{aligned}$$

where $[2, n] = \{2, \ldots, n\}$.

Since by assumption $a_1 \leqslant a_i$ for all i, we get

$$a_i\Bigg(\sum_{\substack{\sigma\subset[2,n]\\ |\sigma|=h-1}} a_\sigma\Bigg) = \sum_{\substack{\sigma\subset[2,n]\\ |\sigma|=h-1,\, i\notin\sigma}} a_i a_\sigma + \sum_{\substack{\sigma\subset[2,n]\\ |\sigma|=h-1,\, i\in\sigma}} a_i a_\sigma$$

$$\geqslant \sum_{\substack{\sigma\subset[2,n]\\ |\sigma|=h-1,\, i\notin\sigma}} a_i a_\sigma + \sum_{\substack{\sigma\subset[2,n]\\ |\sigma|=h-1,\, i\in\sigma}} a_1 a_\sigma.$$

Thus,

$$\frac{1}{(h-1)!}\prod_{i=1}^{h}(a_i + a_{i+1} + \cdots + a_n)$$

$$\geqslant \sum_{\substack{\sigma\subset[2,n]\\ |\sigma|=h-1}} a_1 a_\sigma + \sum_{i=2}^{n}\Bigg(\sum_{\substack{\sigma\subset[2,n]\\ |\sigma|=h-1,\, i\in\sigma}} a_1 a_\sigma\Bigg) + \sum_{i=2}^{n}\Bigg(\sum_{\substack{\sigma\subset[2,n]\\ |\sigma|=h-1,\, i\notin\sigma}} a_i a_\sigma\Bigg)$$

$$= h\Bigg(\sum_{\substack{\sigma\subset[n]\\ |\sigma|=h,\, 1\in\sigma}} a_\sigma\Bigg) + h\Bigg(\sum_{\substack{\sigma\subset[n]\\ |\sigma|=h,\, 1\notin\sigma}} a_\sigma\Bigg) = h\Bigg(\sum_{\substack{\sigma\subset[n]\\ |\sigma|=h}} a_\sigma\Bigg),$$

as desired. □

2. Antichains

In this section, we introduce the terminology of antichains and their connection with squarefree monomial ideals. For two sets A, B, we denote by $A \setminus B$ the set $\{x\colon\ x \in A,\ x \notin B\}$.

Definition 2.1. An *antichain* on a vertex set V is a collection $\mathcal{A}$ of subsets of V such that $A \not\subset A'$ for any two distinct sets $A, A' \in \mathcal{A}$.

Let J be an arbitrary squarefree monomial ideal in the variables $x_1, \ldots, x_n$ and $J = \bigcap_{i=1}^{r} J_i$ its presentation as intersection of its irreducible components. Each J_i is generated by a subset of the variables $x_1, \ldots, x_n$. We assign to each J_i the set $A_i = \{j \in [n]\colon\ x_j \in J_i\}$ and put $\mathcal{A}_J = \{A_i \subset [n]\colon\ i = 1, \ldots, r\}$. The hypergraph $\mathcal{A}_J$ has the property that there is no inclusion between the sets belonging to $\mathcal{A}_J$. Note that $\mathcal{A}_J$ is an ordinary graph if all elements in $\mathcal{A}_J$ have cardinality 2. Thus, the hypergraph $\mathcal{A}_J$ is an antichain on the vertex set $[n]$. We call $\mathcal{A}_J$ the *antichain of J*.

Definition 2.2. Let $\mathcal{A}$ be a antichain on the vertex set $[n]$. A subset $G \subset [n]$ is called a *generator of* $\mathcal{A}$, if $G \cap A \neq \emptyset$ for all $A \in \mathcal{A}$. A generator G is called *minimal*, if no proper subset of G is a generator.

We note that $G \subset [n]$ is a minimal generator of $\mathcal{A}_J$ if and only if $\prod_{i \in G} x_i \in G(J)$.

Definition 2.3. A subset $D \subset [n]$ is called *completely disconnected* with respect to $\mathcal{A}$, if no element of $\mathcal{A}$ is contained in D.

Remark 2.4. A subset D of $[n]$ is completely disconnected with respect to an antichain $\mathcal{A}$ if and only if $[n] \setminus D$ is a generator of $\mathcal{A}$. Thus, maximally completely disconnected sets correspond to minimal generators.

If all elements in $\mathcal{A}$ have cardinality 2, $\mathcal{A}$ is an ordinary graph. In this case, D being completely disconnected simply means that no two points in D are connected by an edge.

Definition 2.5. The *height* $h(\mathcal{A})$ of an antichain $\mathcal{A}$ is the minimum of the cardinality of the sets in it.

Definition 2.6. The *multiplicity* $e(\mathcal{A})$ of $\mathcal{A}$ is the number of elements in $\mathcal{A}$ of cardinality $h(\mathcal{A})$.

With this, for a squarefree monomial ideal J the height of the antichain $\mathcal{A}_J$ is the same as the height of J and the multiplicity of $\mathcal{A}_J$ is the same as the multiplicity of S/J.

Given an antichain $\mathcal{A}$ on the vertex set $[n]$, we define $G(\mathcal{A})$ to be the set of all minimal generators of $\mathcal{A}$. Then, $G(\mathcal{A})$ is a set of subsets of $[n]$ such that

(i) $G \in G(\mathcal{A})$ if and only if A intersects all sets in $\mathcal{A}$ non-trivially.
(ii) If B is a subset of $[n]$ which intersects all sets in $\mathcal{A}$ non-trivially, then there exists an $A \in G(\mathcal{A})$ such that $A \subset B$.

So, $G(\mathcal{A})$ is also an antichain.

Examples 2.7. The following are some examples to illustrate the connection between the antichains and squarefree monomials:

(1) Consider, for example, the antichain $\mathcal{A}$ of all subsets of $[n]$ of cardinality h. Then $h(\mathcal{A}) = h$, $e(\mathcal{A}) = \binom{n}{n-h}$ and $G(\mathcal{A})$ is the antichain of all subsets of $[n]$ of cardinality $n - h + 1$. The squarefree monomial ideal associated with this antichain is the ideal generated by all sqaurefree monomials of height $n - h + 1$.
(2) Let $I = (x_1, x_2) \cap (x_3, x_2, x_4) \cap (x_2, x_5, x_4)$ be a square free monomial ideal. Then $\mathcal{A}_I$ is the antichain on the vertices $\{1, 2, 3, 4, 5\}$ consisting of the subsets $\{1, 2\}$, $\{2, 3, 4\}$, $\{2, 4, 5\}$. We have $h(\mathcal{A}) = 2 = h(I)$, $G(\mathcal{A}) = \{\{2\}, \{1, 4\}, \{1, 3, 5\}\}$, and $e(\mathcal{A}) = 1$.
(3) Let $I = (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4)$. Then I is of height 2 and the multiplicity of R/I is 3. We have $\mathcal{A} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$ and $G(\mathcal{A}) = \{\{1, 2, 4\}, \{1, 3\}, \{2, 3\}, \{2, 4\}\}$.
(4) Let $I = (x_1, x_2) \cap (x_1, x_3) \cap (x_1, x_4) \cap (x_1, x_5) \cap (x_2, x_3, x_5)$. Then I has codimension 2 and multiplicity 4 and $\mathcal{A} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3, 5\}\}$. We will look at this example again in Section 4.

Lemma 2.8. *Let $\mathcal{A}$ be an antichain, $A \in \mathcal{A}$, and $x \in A$. Then there exists an element $G \in G(\mathcal{A})$ such that $G \cap A = \{x\}$.*

Proof. Let $G_0 = \bigcup_{A' \in \mathcal{A}} (A' \setminus A)$. Then $G_1 = G_0 \cup \{x\}$ is a generator of $\mathcal{A}$. Let $G \subset G_1$ be a minimal generator of $\mathcal{A}$. We claim that $G \cap A = \{x\}$. For this it remains to show that $x \in G$. Suppose $x \notin G$. Then $G \subset G_0$ and so $G \cap A = \emptyset$, a contradiction. □

Theorem 2.9. *If $\mathcal{A}$ is an antichain, then $G(G(\mathcal{A})) = \mathcal{A}$.*

Proof. Let $A \in \mathcal{A}$. By definition of the minimal generating set, $A \cap G \neq \emptyset$ for any $G \in G(\mathcal{A})$. By Lemma 2.8, if $x \in A$, then there exists $G \in G(\mathcal{A})$ such that $G \cap A = \{x\}$. So, A is a minimal generator for $G(\mathcal{A})$.

Conversely, if $A \in G(G(\mathcal{A}))$, then $A \cap G \neq \emptyset$ for any $G \in G(\mathcal{A})$. Suppose A does not contain any of the finite number of sets in $\mathcal{A}$. Then we can pick $c \in C$ for each $C \in \mathcal{A}$ that is not in A. The set containing these elements c meets each of the sets in the antichain $\mathcal{A}$ and hence contains a minimal generator G of $\mathcal{A}$. However, its intersection with A is empty, contradicting the fact $A \in G(G(\mathcal{A}))$. So, there exists $B \in \mathcal{A}$ such that $B \subset A$. Since B is a generator of $G(\mathcal{A})$, this containment must be an equality and hence $A \in \mathcal{A}$. This finishes the proof. □

Remark 2.10. It is easy to see that Theorem 2.9 is equivalent to the well-known Alexander duality for simplicial complexes.

Lemma 2.8 and Theorem 2.9 imply the following.

Corollary 2.11. *For each minimal generator G of $\mathcal{A}$ and any $x \in G$, there exists $A \in \mathcal{A}$ such that $G \cap A = \{x\}$.*

3. The Taylor bound

The reader may wonder why we replaced inequality (1) by the weaker inequality in Theorem 1.1. The reason is, that there is a relationship of the inequality in Theorem 1.1 to the following conjecture.

Conjecture 3.1 [2]. Let $I \subset S$ be graded ideal of codimension h with graded Betti numbers $\beta_{ij} = \dim_K \operatorname{Tor}_i^S(K, S/I)_j$ and let $M_i = \max\{j\colon \beta_{ij} \neq 0\}$ for $i = 1, \ldots, n$. Then

$$e(S/I) \leqslant \frac{1}{h!} \prod_{i=1}^{h} M_i.$$

In order to see how these two inequalities are related to each other, let I be a monomial ideal with $G(I) = \{f_1, \ldots, f_m\}$. For $\sigma \subset [n]$, $\sigma = \{j_1, \ldots, j_k\}$, we let $f_\sigma =$

$\operatorname{lcm}(f_{j_1}, \dots, f_{j_k})$ be the least common multiple of $f_{j_1}, \dots, f_{j_k}$ and set

$$L_i = \max\bigl\{\deg f_\sigma\colon\ \sigma \subset [n],\ |\sigma| = i\bigr\} \quad \text{for } i = 1, \dots, m.$$

Note that L_i is the highest degree of a generator of T_i, where T is the Taylor complex of the ideal I, cf. [4]. Comparing the Taylor complex with the graded minimal free S-resolution of S/I, we see that

$$M_i \leqslant L_i \quad \text{for } i = 1, \dots, n.$$

In particular, the following conjecture is weaker than Conjecture 3.1.

Conjecture 3.2. Let I be a monomial ideal of codimension h. Then

$$e(S/I) \leqslant \frac{1}{h!} \prod_{i=1}^{h} L_i.$$

We say the Taylor bound holds for I if Conjecture 3.2 is true for S/I.

One should remark, that in most cases the Taylor bound will be stronger than the bound given in Theorem 1.1.

Theorem 3.3. *With the hypotheses of Theorem* 1.1, *there exist monomials $g_i \in I$ such that*

$$\operatorname{lcm}(g_1, \dots, g_i) = \sum_{j=h-i+1}^{n} a_j \quad \textit{for } i = 1, \dots, h.$$

In particular, if $\{g_1, \dots, g_h\} \subset G(I)$, then the Taylor bound holds for I.

For the proof of Theorem 3.3 it is convenient to first polarize the ideal I. Recall the following definitions and facts about polarization: let $f = x_1^{a_1} \cdots x_n^{a_n}$; the *polarization of* f is the squarefree monomial $f^p = \prod_{i=1}^{n} \prod_{j=1}^{a_i} x_{ij}$ in the new set of variables

$$\{x_{11}, \dots, x_{1a_1}, x_{21}, \dots, x_{2a_2}, \dots, x_{n1}, \dots, x_{na_n}\}.$$

Suppose that $G(I) = \{f_1, \dots, f_m\}$. The *polarization of I* is the squarefree monomial ideal $I^p = (f_1^p, \dots, f_m^p)$ in the polynomial ring S in as many variables x_{ij} as are needed to polarize the generators of I.

The most important fact about polarization is that S/I^p is a deformation of S/I, so that for all i and j, the ijth graded Betti number of S/I^p and of S/I is the same. This implies, in particular, that S/I^p and S/I have the same multiplicity and the same codimension.

Let S_1 and S_2 be polynomial rings over K. We say that the ideals $I_1 \subset S_1$ and $I_2 \subset S_2$ are *equivalent*, if there exists a common polynomial ring extension S of S_1 and S_2 such that $I_1 S = I_2 S$. We write $I_1 \sim I_2$, if I_1 and I_2 are equivalent.

Let I and J be monomial ideals. Then

$$(I \cap J)^p \sim I^p \cap J^p. \tag{2}$$

For the convenience of the reader we sketch the simple proof of (2). We first note that if L is a monomial ideal (not necessarily minimally) generated by $h_1, \dots, h_r$, then $L^p \sim (h_1^p, \dots, h_r^p)$.

Let $G(I) = \{f_1, \dots, f_r\}$ and $G(J) = \{g_1, \dots, g_s\}$, and let $\operatorname{lcm}(f, g)$ denote the least common multiple of two monomials f and g. Then

$$I \cap J = \{\operatorname{lcm}(f_i, g_j)\colon\ i = 1, \dots, r,\ j = 1, \dots, s\}.$$

In general, of course, the elements $\operatorname{lcm}(f_i, g_j)$ do not form a minimal set of generators of $I \cap J$. Nevertheless, since $\operatorname{lcm}(f, g)^p = \operatorname{lcm}(f^p, g^p)$, it follows that

$$\begin{aligned} I^p \cap J^p &= \{\operatorname{lcm}(f_i^p, g_j^p)\colon\ i = 1, \dots, r,\ j = 1, \dots, s\} \\ &= \{\operatorname{lcm}(f_i, g_j)^p\colon\ i = 1, \dots, r,\ j = 1, \dots, s\} \sim (I \cap J)^p. \end{aligned}$$

As a consequence of these considerations, we have

Proposition 3.4. *The Taylor bound holds for I if and only if it holds for I^p.*

Proof of Theorem 3.3. Let $I = \bigcap_i I_i$ where the I_i are the irreducible components of I. Then $I^p \sim \bigcap_i I_i^p$. Suppose that $I_i = (x_{i_1}^{b_1}, \dots, x_{i_k}^{b_k})$. Then

$$I_i^p = \left(\prod_{j=1}^{b_1} x_{i_1 j}, \dots, \prod_{j=1}^{b_k} x_{i_k j}\right) = \bigcap_{j_1, \dots, j_k} (x_{i_1 j_1}, \dots, x_{i_k j_k}),$$

where in this intersection is taken over all j_i with $1 \leqslant j_i \leqslant b_i$ for $i = 1, \dots, k$. Thus, we see that the irreducible components of I^p are all the form $(x_{i_1 j_1}, \dots, x_{i_k j_k})$ with $i_1 < i_2 < \dots < i_k$.

Let $A_i = \{(i, j)\colon\ j = 1, \dots, a_i\}$. The antichain of I^p is defined on the set $A = \bigcup_{i=1}^n A_i$. Notice that A is the disjoint union of the A_i and that each of the A_i is completely disconnected. Let $i_1, \dots, i_{n-h+1}$ be a subset of $[n]$ with $i_1 < i_2 < \dots < i_{n-h+1}$ and set $G = \bigcup_{j=1}^{n-h+1} A_{i_j}$. Then since each $S \in \mathcal{A}_I$ has cardinality at least h and $|A_i \cap S| \leqslant 1$ for each i, we see that no $S \in \mathcal{A}_I$ is contained in $A \setminus G$ (which is the disjoint union of $h - 1$ of the sets A_i). By the preceding remarks, this implies that G is a generator. The monomial in I^p corresponding to G is $\prod_{j=1}^{n-h+1} (\prod_{k=1}^{a_{i_j}} x_{i_j k})$. Specializing to I, we see that $\prod_{j=1}^{n-h+1} x_{i_j}^{a_{i_j}} \in I$. In particular,

$$g_i = \prod_{j=h-i+1}^{n-i+1} x_j^{a_j} \in I \quad \text{for } i = 1, \dots, h \quad \text{and} \quad \operatorname{lcm}(g_1, \dots, g_i) = \sum_{j=n-i+1}^{n} a_j. \qquad \square$$

An ordinary graph is called bipartite if the vertex set can be written as a disjoint union of two sets S_1, S_2 each of which is completely disconnected. It is a complete bipartite graph if in addition each vertex in S_1 is joined to each vertex in S_2 by an edge. We will generalize this by saying that an antichain $\mathcal{A}$ is complete multipartite if the vertex set $[n]$ can be written as a disjoint union of a finite number of sets S_i each of which is completely disconnected with respect to the antichain $\mathcal{A}$ and for each i, j with $i \neq j$, and each vertex x in S_i there exists a set $S \in \mathcal{A}$ such that $S \cap S_i = \{x\}$ and $S \subset S_i \cup S_j$. In other words, the complement of each S_i of a complete multipartite antichain is a minimal generator. Then Theorem 3.3 simply proves the Taylor bound for monomial ideals whose antichains are completely multipartite.

With the same methods as in the proof of Theorem 3.3, we can show

Theorem 3.5. *Let $I \subset S$ be a squarefree monomial ideal of codimension h and assume that $[n]$ can be written as a disjoint union $[n] = \bigcup_{i=1}^{r} A_i$ such that each A_i is completely disconnected with respect to $\mathcal{A}_I$. Let $|A_i| = a_i$ for $i = 1, \dots, r$ and assume that $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r$. Then*

$$e(S/I) \leqslant \frac{1}{h!} \prod_{i=1}^{h} (a_i + a_2 + \cdots + a_r).$$

Assume, in addition, that for each $c \in A_i$ there exists a subset $\{j_1, \dots, j_{h-1}\} \in [r] \setminus \{i\}$ and $c_{j_k} \in A_{j_k}$ such that $\{c, c_{j_1}, \dots, c_{j_{h-1}}\}$ belongs to $\mathcal{A}_I$. Then the Taylor bound holds for I.

Proof. Let $A \in \mathcal{A}_I$ with $A = \{j_1, \dots, j_h\}$. By hypothesis, this means that there exists a subset $\{i_1, \dots, i_h\}$ of $[r]$ such that $j_k \in A_{i_k}$, for $1 \leqslant k \leqslant h$. Thus, we conclude that there exist at most $\sum_{1 \leqslant i_1 < i_2 < \cdots < i_h \leqslant r} a_{i_1} a_{i_2} \cdots a_{i_h}$ sets $A \in \mathcal{A}_I$ of cardinality h. Since this number is the multiplicity of S/I, the same arguments as in the proof of Theorem 1.1 yield the desired inequality.

The additional assumption implies that for all subsets $\{i_1, \dots, i_{r-h+1}\} \subset [r]$ of cardinality $r - h + 1$, the set $G = \bigcup_{k=1}^{r-h+1} A_{i_k}$ is a minimal generator. Thus, an argument as in Theorem 3.3 concludes the proof. □

4. Proof of Taylor bound in codimension two

Throughout this section $\mathcal{A}$ is an antichain on the vertex set $[n]$. Let $G(\mathcal{A}) = \{G_1, \dots, G_m\}$. For a subset I of $[m]$, as in the introduction, we define $G_I = \bigcup_{i \in I} G_i$ and let $L_i = \max\{|G_I|: |I| = i\}$.

Lemma 4.1. *Let the height of $\mathcal{A}$ be 2. Then vertex set $[n]$ on the antichain $\mathcal{A}$ can be written as a disjoint union of a finite number of subsets S_i, $0 \leqslant i \leqslant t$ with $|S_i| = a_i$ such that*

(a) *Each S_i is completely disconnected with respect to $\mathcal{A}$;*
(b) $L_1 = \sum_{i=2}^{i=t} a_i$;

(c) $L_2 \geqslant \sum_{i=1}^{i=t} a_i$;
(d) *$S_1 \cup S_0$ and $S_2 \cup S_0$ are completely disconnected with respect to $\mathcal{A}$*;
(e) *For each i and j such that $2 < i < j$ and each $x \in S_j$ there exists an $A \in \mathcal{A}$ with $A \cap S_j = \{x\}$ and $A \subset S_i \cup S_j$*;
(f) *For all $i \geqslant 1$ and all $x \in S_i$ there exists $A \in \mathcal{A}$ such that $A \cap S_i = \{x\}$.*

Proof. First, we will construct the decomposition of $[n]$ into disjoint subsets S_i. Let A_1 be a maximal element in $G(\mathcal{A})$ and let A_2 be maximal among the elements in $G(\mathcal{A})$ whose union with A_1 will be maximal. Let $S_0 = [n] \setminus (A_1 \cup A_2)$. Let $S_1 = A_2 \setminus A_1$, $S_2 = A_1 \setminus A_2$, and $T_2 = A_1 \cap A_2$. Clearly, S_i, $i = 0, 1, 2$, and T_2 are all pairwise disjoint and their union is $[n]$. Further, since A_1 is a generator and $S_1 \cup S_0$ is in its complement, it is completely disconnected with respect to $\mathcal{A}$. Similarly, $S_2 \cup S_0$ is completely disconnected with respect to $\mathcal{A}$.

Now, consider the sub-antichain $\mathcal{A}_3$ of $\mathcal{A}$ consisting of all the subsets of $\mathcal{A}$ contained in T_2.

If $\mathcal{A}_3 = \emptyset$, then we let $t = 3$ and $S_3 = T_2$. If $\mathcal{A}_3 \neq \emptyset$, then let T_3 be a minimal generator for $\mathcal{A}_3$. Suppose $T_3 = T_2$ and let $x \in T_2$. Then $T_2 \setminus \{x\}$ is not a generator with respect to $\mathcal{A}_3$. That is, there exists an element $A \in \mathcal{A}$ with $A \subset T_2$ and $(T_2 \setminus \{x\}) \cap A = \emptyset$. But this implies that $A = \{x\}$, a contradiction since height $\mathcal{A}$ is 2. So, $T_3 \neq T_2$. Let $S_3 = T_2 \setminus T_3$. Then S_3 is completely disconnected with respect to $\mathcal{A}$ and T_3 is strictly smaller than T_2. Proceed with T_3 now by letting $\mathcal{A}_4$ the sub-antichain of $\mathcal{A}$ consisting of all the subsets in $\mathcal{A}$ contained in T_3. If $\mathcal{A}_4 = \emptyset$, we let $t = 4$ and $S_4 = T_3$. Otherwise we proceed as before. Since $[n]$ is finite, this process will stop. We get $[n] = \bigcup_{i=0}^{i=t} S_i$.

Now, by construction, each S_i is completely disconnected with respect to $\mathcal{A}$. Also, $A_1 = \bigcup_{i=2}^{t} S_i$ and $A_2 = S_1 \cup (\bigcup_{i=3}^{t} S_i)$. So, we get (b) and (c) to be true. Assertion (d) follows from the construction of S_i, $i = 0, 1, 2$.

It remains to prove (e) and (f).

For (e), we remark that in our construction,

$$T_i = \begin{cases} \bigcup_{j>i} S_j, & \text{for } 2 \leqslant i < t, \\ S_t, & \text{for } i = t. \end{cases}$$

Now, let $j > i > 2$ and $x \in S_j$. Then $i \neq t$ and T_i is a minimal generator for the sub-antichain $\mathcal{A}_i$. So, there exists $A \in \mathcal{A}_i$, such that $A \cap T_i = A \cap S_j = \{x\}$. But $A \in \mathcal{A}_i$, $i > 2$ implies that $A \subset T_{i-1} = T_i \cup S_i$ and so $A \subset S_i \cup S_j$. This finishes the proof of (e).

To prove (f), we note that $S_i \subset A_1$ for $i \geqslant 2$ and $S_1 \subset A_2$. So Corollary 2.11 implies the assertion. □

Now, we note that the multiplicity $e(\mathcal{A})$ is the number of elements in $\mathcal{A}$ containing exactly h elements where h is the smallest cardinality for an element in $\mathcal{A}$. Recall from Section 2 that this minimal cardinality of an element in $\mathcal{A}$ is called the height of $\mathcal{A}$.

Theorem 4.2. *Let the height of $\mathcal{A}$ be* 2. *Then* $2e(\mathcal{A}) \leqslant L_1 L_2$.

Proof. Let $V = \bigcup_{i=0}^{t} S_i$ be the partition of $[n]$ as in the previous proposition. Then $L_1 L_2 \geqslant \sum_{i=1}^{t} a_i \sum_{i=2}^{t} a_i$. Let

$$\tau = \tau(\mathcal{A}) = 2e(\mathcal{A}) - \sum_{i=1}^{t} a_i \sum_{i=2}^{t} a_i.$$

We would like to estimate $e(\mathcal{A})$ and show that τ is not positive.

Since the $\mathcal{A}$ has height 2, we need to count only the subsets in $\mathcal{A}$ with exactly two elements. Since each S_i is completely disconnected with respect to our antichain, there can be at most $\sum_{i>j>2} a_i a_j$ sets in $\mathcal{A}$ of size 2 contained in $\bigcup_{i>2} S_i$. There can be at most $a_1 a_2$ such sets in $S_1 \cup S_2$. Recall that by Lemma 4.1(d), $S_1 \cup S_0$ and $S_2 \cup S_0$ are both completely disconnected. So, for $i > 2$, there can be at most $a_i(a_1 + a_0)$ sets of size 2 that can be in $S_0 \cup S_1 \cup S_i$ and at most $a_i a_2$ sets of size 2 in $S_2 \cup S_i$. Thus, as a first approximation, from Lemma 4.1(d)–(f), we get

$$e(\mathcal{A}) \leqslant \sum_{t \geqslant i > j \geqslant 3} a_i a_j + a_1 a_2 + (a_1 + a_2 + a_0) \sum_{i \geqslant 3} a_i. \tag{3}$$

This is a crude approximation. For each $i \geqslant 3$, since S_i is completely disconnected with respect to $\mathcal{A}$, we must have a minimal generator in its complement. Call it G_i. By Lemma 4.1(e), for each $j > i$ and each $x \in S_j$ there exists $A \in \mathcal{A}$ such that $A \cap S_j = \{x\}$ and $A \subset S_i \cup S_j$. Since $G_i \cap S_i = \emptyset$ but $G_i \cap A \neq \emptyset$, it follows that $G_i \cap A = \{x\}$. However, since $x \in S_j$ was arbitrarily chosen, we see that $S_j \subset G_i$. So, G_i must contain $\bigcup_{j>i} S_j$.

Let $X_i = G_i \cap (S_1 \cup S_0)$, let $r_i = |X_i|$, and let $Y_i = G_i \setminus X_i$.

Recall that $A_1 = \bigcup_{i \geqslant 2} S_i$. So,

$$|G_i \cup A_1| = \left| G_i \bigcup_{i \geqslant 2} S_i \right| = r_i + \sum_{i=2}^{t} a_i \leqslant \sum_{i=1}^{t} a_i,$$

by the choice of A_2. Thus $r_i \leqslant a_1$.

Since the complement of the generator G_i must be completely disconnected, for $i \geqslant 3$ there can be no $A \in \mathcal{A}$ of size 2 such that $A \subset S_i \cup [(S_1 \cup S_0) \setminus X_i]$. So, we may subtract $\sum_{i \geqslant 3} a_i(a_1 + a_0 - r_i)$ from our estimate of e in Eq. (3).

Further, let write $Z_i = \sum_{2 \leqslant j < i} S_j \setminus Y_i$. Then Z_i is in the complement of G_i. There is no $A \in \mathcal{A}$, $A = \{x, y\}$ with $x \in Z_i$ and $y \in S_i$, because for such A, we would have that $A \cap G_i = \emptyset$, a contradiction since G_i is a generator. Thus, if $z_i = |Z_i|$, we may further subtract $\sum_{i \geqslant 3} a_i z_i$ from our estimate of e in Eq. (3).

Moreover, since A_1 is maximal among the minimal generators of $\mathcal{A}$,

$$|G_i| = r_i + \sum_{j>i} a_j \sum_{2 \leqslant j < i} a_j - z_i \leqslant \sum_{j \geqslant 2} a_j.$$

We get $r_i \leqslant a_i + z_i$. So,

$$e \leqslant \sum_{t\geqslant i>j\geqslant 3} a_i a_j + a_1 a_2 + (a_1 + a_2 + a_0)\sum_{i\geqslant 3} a_i - \sum_{i\geqslant 3} a_i(a_1 + a_0 - r_i + z_i).$$

Thus,

$$e \leqslant \sum_{i>j\geqslant 3} a_i a_j + a_1 a_2 + \sum_{i\geqslant 3} a_i(a_2 + r_i - z_i).$$

Notice that

$$L_1 L_2 \geqslant \sum_{i\geqslant 2} a_i \sum_{i\geqslant 1} a_i = 2\sum_{i>j\geqslant 3} a_i a_j + \sum_{i\geqslant 3} a_i(2a_2 + a_1 + a_i) + a_1 a_2 + a_2^2.$$

Thus,

$$2e - \sum_{i\geqslant 2} a_i \sum_{i\geqslant 1} a_i \leqslant \sum_{i\geqslant 3} a_i(2r_i - 2z_i - a_1) + a_1 a_2 - \sum_{i\geqslant 2} a_i^2.$$

But $a_1 = |S_1| \leqslant |S_2| = a_2$ follows from $|A_2| \leqslant |A_1|$. Hence, $a_1 a_2 \leqslant a_2^2$ and we get

$$\tau \leqslant \sum_{i\geqslant 3} a_i(2r_i - 2z_i - a_1 - a_i).$$

Now,

$$2r_i - 2z_i - a_i - a_1 = (r_i - z_i - a_i) + (r_i - a_1) - z_i.$$

Since $r_i \leqslant a_1$ and $r_i \leqslant a_i + z_i$, this expression is not positive. So, τ is not positive and this finishes the proof. □

Corollary 4.3. *For a monomial ideal of height two, the Taylor bound holds.*

Proof. By Proposition 3.4 we may assume that I is squarefree. By our definition, the antichain $\mathcal{A}_I$ of I is such that $e(\mathcal{A}_I) = e(I)$ and $L_i(\mathcal{A}_I) = L_i(I)$ for all i. By Theorem 4.2 applied to the antichain of I, we get $2e(S/I) \leqslant L_1 L_2$. □

For an ideal I in a Noetherian ring, $\sup \operatorname{height}(I)$ is the maximal height of a minimal prime ideal of I.

The next result follows from Theorem 4.2 and the fact that $G(G(\mathcal{A})) = \mathcal{A}$ for any antichain $\mathcal{A}$, see Theorem 2.9.

Corollary 4.4. *Let S be a polynomial ring in n variables and I a squarefree monomial ideal generated by elements of degree* 2 *and higher. Then the number of generators of I in degree* 2 *is bounded by $n(\sup \operatorname{height}(I))/2$.*

Example 4.5. Let $I = (x_1, x_2) \cap (x_1, x_3) \cap (x_1, x_4) \cap (x_1, x_5) \cap (x_2, x_3, x_5)$. Then I has codimension 2 and multiplicity 4 and we have $\mathcal{A} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3, 5\}\}$, $A_1 = \{2, 3, 4, 5\}$, and $A_2 = \{1, 2\}$. Hence, $S_1 = \{1\}$, $S_2 = \{3, 4, 5\}$, $S_0 = \emptyset$, $S_3 = \{2\}$, and $2e = 8 \leqslant 4 \times 5 = 20$.

Example 4.6. Let $\mathcal{A} = \{\{1, 3\}, \{1, 6\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 7\}, \{7, 2\}\}$. Then $A_1 = \{2, 3, 5, 6\}$, $A_2\{3, 4, 6, 7\}$. Hence, $S_1 = \{4, 7\}$, $S_2 = \{2, 5\}$, $S_0 = \{1\}$, and $S_3 = \{3, 6\}$, and $2e = 16 \leqslant 4 \times 6 = 24$.

5. Almost complete intersections

Let I be a monomial ideal of height h. It is called an almost complete intersection if $|G(I)| = h + 1$. In this section, we show that the Taylor bound holds for almost complete intersection monomial ideals.

Theorem 5.1. *Let I be an almost complete intersection of height $h \geqslant 2$. Let $G(I) = \{f_1, \ldots, f_{h+1}\}$. Then after a suitable renumbering of the elements of $G(I)$ either $f_1, \ldots, f_h$ is a regular sequence, or the height of $J = (f_{h-1}, f_h, f_{h+1})$ is 2, and $f_1, \ldots, f_{h-2}$ is a regular sequence modulo J.*

Proof. Polarizing I we may assume that I is a squarefree monomial ideals. If $h = 3$, then there is nothing to prove. Now suppose that $h > 3$. We consider the graph G whose vertices are the elements $\{1, \ldots, h + 1\}$ and for $i < j$, (i, j) is an edge of G if f_i and f_j have a common factor.

The graph G has the property that any two edges have a common vertex. In fact, suppose that (i, j) and (k, l) are two edges with $\{i, j\} \cap \{k, l\} = \emptyset$. Then there exist x_r and x_s such that x_r divides f_i and f_j and x_s divides f_k and f_l. Then $I \subset \{f_m\colon m \neq i, j, k, l\} + (x_r, x_s)$ and so height$(I) \leqslant h - 1$, a contradiction.

Let k be the number of edges of G. We have $k \geqslant 1$, otherwise height$(I) = h + 1$, a contradiction.

If $k = 1$, we may assume that $(h, h+1)$ is the only edge of G. Then height$(f_h, f_{h+1}) = 1$ and $f_1, \ldots, f_{h-1}$ is a regular sequence modulo (f_h, f_{h+1}). Therefore, $J = (f_{h-1}, f_h, f_{h+1})$ is of height 2 and $f_1, \ldots, f_{h-1}$ is a regular sequence modulo J and we are done.

If $k = 2$, the two edges must have a common vertex and we may assume that $(h-1, h+1)$ and $(h, h+1)$ are these two edges. It follows that height$(J) \leqslant 2$. Suppose that height$(J) = 1$, then $I = (f_1, \ldots, f_{h-2}, J)$ can have height at most $h - 1$, a contradiction. Therefore, height$(J) = 2$ and $f_1, \ldots, f_{h-2}$ is a regular sequence modulo J, since there are no other edges in G.

If $k \geqslant 3$, we may assume that two of the edges are $(h - 1, h + 1)$ and $(h, h + 1)$. Since the other edges (i, j) have to have a vertex in common with $(h - 1, h + 1)$ and $(h, h + 1)$, we either have (1) $j = h + 1$ for all other edges, or (2) $j = h$ or $j = h - 1$ for some edge. After renumbering, we may assume that $j = h$. Then in the second case, i must be $h - 1$, because otherwise (i, h) has no vertex in common with $(h - 1, h + 1)$. So in case (2), we have the edges $(h-1, h+1)$, $(h-1, h)$, and $(h, h+1)$, and there can be no other edges in the graph.

The monomial ideal generated by $x_1y_1, \dots, x_hy_h, x_1x_2\cdots x_h$ is an example for the first case, while the monomial ideal with generators $x_1y_1, x_2y_2, \dots, x_{h-1}y_{h-1}, x_{h-1}y_h, x_hy_h$ an example for the second case.

In the first case, $f_1, \dots, f_h$ is a regular sequence, in the second case, $f_1, \dots, f_{h-2}$ is a regular sequence modulo $J = (f_{h-1}, f_h, f_{h+1})$. □

Theorem 5.2. *Let I be an almost complete intersection monomial ideal. Then the Taylor bound holds for I.*

Proof. Let I be the monomial almost complete intersection ideal with $G(I) = \{f_1, \dots, f_{h+1}\}$ and $d_i = \deg f_i$ for $i = 1, \dots, h+1$. By Theorem 5.1, there are two types of almost complete intersections. In the first case, $f_1, \dots, f_h$ is a regular sequence. Let J be the ideal generated by $f_1, \dots, f_h$. Since $\operatorname{height}(J) = \operatorname{height}(I) = h$ it follows that $e(S/I) \leqslant e(S/J)$. On the other hand, we obviously have $L_i(J) \leqslant L_i(I)$ for $i = 1, \dots, h$. Thus, it suffices to show that the Taylor bound holds for J. But this follows from Theorem 1.1.

Dealing with the second type of monomial almost complete intersections, we may assume that the first $h-1$ generators, f_i, $1 \leqslant i \leqslant h-1$ of the ideal I form a regular sequence and the last two generators, f_h and f_{h+1}, have a common divisor with f_{h-1} and are relatively prime to f_i for $i = 1, \dots, h-2$. We can take f_{h-1} to be of the highest degree among the last three. We can also renumber $f_1, \dots, f_{h-2}$ so that $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_{h-2}$. By Theorem 4.2, we have

$$e\big(S/(f_{h-1}, f_h, f_{h+1})\big) \leqslant d_{h-1}\frac{d_{h-1}+x}{2}.$$

Now, we distinguish two cases. In the first case, we assume that $2d_1 \leqslant d_{h-1} + x$. Let $N = (f_2, \dots, f_{h+1})$. Then N is an almost complete intersection of height $h-1$, so that we can apply induction. Notice that $L_i(N) \leqslant L_i(I)$, $1 \leqslant i \leqslant h-1$. Also, $L_h(I) = d_1 + d_2 + \cdots + d_{h-2} + d_{h-1} + x$.

Now,

$$e(S/I) = d_1 e(S/N) \leqslant d_1 \frac{1}{(h-1)!} \prod_{i=1}^{i=h-1} L_i(N).$$

But, $hd_1 = (h-2)d_1 + 2d_1 \leqslant d_1 + d_2 + \cdots + d_{h-2} + (d_{h-1} + x)$. Hence, $hd_1 \leqslant L_h(I)$. Putting these inequalities together, we see that

$$e(S/I) \leqslant \frac{1}{h!}\prod_{i=1}^{h} L_i,$$

as desired.

In the second case, we assume that $2d_1 > d_{h-1} + x$. We have

$$e(S/I) = \prod_{i=1}^{h-2} d_i e\bigl(S/(f_{h-1}, f_h, f_{h+1})\bigr) \leqslant \prod_{i=1}^{h-1} d_i \frac{(d_{h-1} + x)}{2}.$$

Since, $L_i(f_1, \dots, f_{h-1}) \leqslant L_i(I)$ and $f_1, \dots, f_{h-1}$ form a regular sequence, we get

$$(h-1)! \prod_{i=1}^{h-1} d_i \leqslant \prod_{i=1}^{h-1} L_i(f_1, \dots, f_{h-1}) \leqslant \prod_{i=1}^{h-1} L_i(I).$$

This in turn means

$$e(S/I) \leqslant \prod_{i=1}^{h-1} d_i \frac{(d_{h-1} + x)}{2} \leqslant \frac{1}{(h-1)!} \prod_{i=1}^{h-1} L_i(I) \frac{d_{h-1} + x}{2}.$$

Finally,

$$\begin{aligned} h(d_{h-1} + x) &= (h-2)(d_{h-1} + x) + 2(d_{h-1} + x) \leqslant (h-2)2d_1 + 2(d_{h-1} + x) \\ &\leqslant 2d_1 + \cdots + 2d_{h-2} + 2(d_{h-1} + x) = 2L_h. \end{aligned}$$

This finishes the proof. □

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